

A Relevant proofs

A.1 Proof of Theorem 2

Proof. The proof is similar to that given by (Sutton et al. 2009) for TDC, but it is based on multi-time-scale stochastic approximation.

For the VMTDC algorithm, a new one-step linear TD solution is defined as:

$$0 = \mathbb{E}[(\phi - \gamma\phi' - \mathbb{E}[\phi - \gamma\phi'])\phi^\top] \mathbb{E}[\phi\phi^\top]^{-1} \mathbb{E}[(\delta - \mathbb{E}[\delta])\phi] = \mathbf{A}^\top \mathbf{C}^{-1}(-\mathbf{A}\boldsymbol{\theta} + \mathbf{b}).$$

The matrix $\mathbf{A}^\top \mathbf{C}^{-1} \mathbf{A}$ is positive definite. Thus, the VMTD's solution is $\boldsymbol{\theta}_{\text{VMTDC}} = \boldsymbol{\theta}_{\text{VMTD}} = \mathbf{A}^{-1} \mathbf{b}$.

First, note that recursion (11) and (12) can be rewritten as, respectively,

$$\boldsymbol{\theta}_{k+1} \leftarrow \boldsymbol{\theta}_k + \zeta_k \mathbf{x}(k),$$

$$\mathbf{u}_{k+1} \leftarrow \mathbf{u}_k + \beta_k \mathbf{y}(k),$$

where

$$\mathbf{x}(k) = \frac{\alpha_k}{\zeta_k} [(\delta_k - \omega_k)\phi_k - \gamma\phi'_k(\phi_k^\top \mathbf{u}_k)],$$

$$\mathbf{y}(k) = \frac{\zeta_k}{\beta_k} [\delta_k - \omega_k - \phi_k^\top \mathbf{u}_k] \phi_k.$$

Recursion (11) can also be rewritten as

$$\boldsymbol{\theta}_{k+1} \leftarrow \boldsymbol{\theta}_k + \beta_k \mathbf{z}(k),$$

where

$$\mathbf{z}(k) = \frac{\alpha_k}{\beta_k} [(\delta_k - \omega_k)\phi_k - \gamma\phi'_k(\phi_k^\top \mathbf{u}_k)],$$

Due to the settings of step-size schedule $\alpha_k = o(\zeta_k)$, $\zeta_k = o(\beta_k)$, $\mathbf{x}(k) \rightarrow 0$, $\mathbf{y}(k) \rightarrow 0$, $\mathbf{z}(k) \rightarrow 0$ almost surely as $k \rightarrow \infty$. That is that the increments in iteration (13) are uniformly larger than those in (12) and the increments in iteration (12) are uniformly larger than those in (11), thus (13) is the fastest recursion, (12) is the second fast recursion and (11) is the slower recursion. Along the fastest time scale, iterations of (11), (12) and (13) are associated to ODEs system as follows:

$$\dot{\boldsymbol{\theta}}(t) = 0, \tag{A-1}$$

$$\dot{\mathbf{u}}(t) = 0, \tag{A-2}$$

$$\dot{\omega}(t) = \mathbb{E}[\delta_t | \mathbf{u}(t), \boldsymbol{\theta}(t)] - \omega(t). \tag{A-3}$$

Based on the ODE (A-1) and (A-2), both $\boldsymbol{\theta}(t) \equiv \boldsymbol{\theta}$ and $\mathbf{u}(t) \equiv \mathbf{u}$ when viewed from the fastest timescale. By the Hirsch lemma (Hirsch 1989), it follows that $\|\boldsymbol{\theta}_k - \boldsymbol{\theta}\| \rightarrow 0$ a.s. as $k \rightarrow \infty$ for some $\boldsymbol{\theta}$ that depends on the initial condition $\boldsymbol{\theta}_0$ of recursion (11) and $\|\mathbf{u}_k - \mathbf{u}\| \rightarrow 0$ a.s. as $k \rightarrow \infty$ for some \mathbf{u} that depends on the initial condition \mathbf{u}_0 of recursion (12). Thus, the ODE pair (A-1)-(refomegavmtdcFastest) can be written as

$$\dot{\omega}(t) = \mathbb{E}[\delta_t | \mathbf{u}, \boldsymbol{\theta}] - \omega(t). \tag{A-4}$$

Consider the function $h(\omega) = \mathbb{E}[\delta | \boldsymbol{\theta}, \mathbf{u}] - \omega$, i.e., the driving vector field of the ODE (A-4). It is easy to find that the function h is Lipschitz with coefficient -1 . Let $h_\infty(\cdot)$ be the function defined by $h_\infty(\omega) = \lim_{r \rightarrow \infty} \frac{h(r\omega)}{r}$. Then $h_\infty(\omega) = -\omega$, is well-defined. For (A-4), $\omega^* = \mathbb{E}[\delta | \boldsymbol{\theta}, \mathbf{u}]$ is the unique globally asymptotically stable equilibrium. For the ODE

$$\dot{\omega}(t) = h_\infty(\omega(t)) = -\omega(t), \tag{A-5}$$

apply $\vec{V}(\omega) = (-\omega)^\top (-\omega)/2$ as its associated strict Liapunov function. Then, the origin of (A-5) is a globally asymptotically stable equilibrium.

Consider now the recursion (13). Let $M_{k+1} = (\delta_k - \omega_k) - \mathbb{E}[(\delta_k - \omega_k) | \mathcal{F}(k)]$, where $\mathcal{F}(k) = \sigma(\omega_l, \mathbf{u}_l, \boldsymbol{\theta}_l, l \leq k; \phi_s, \phi'_s, r_s, s < k)$, $k \geq 1$ are the sigma fields generated by $\omega_0, \mathbf{u}_0, \boldsymbol{\theta}_0, \omega_{l+1}, \mathbf{u}_{l+1}, \boldsymbol{\theta}_{l+1}, \phi_l, \phi'_l, 0 \leq l < k$. It is easy to verify that $M_{k+1}, k \geq 0$ are integrable random variables that satisfy $\mathbb{E}[M_{k+1} | \mathcal{F}(k)] = 0, \forall k \geq 0$. Because ϕ_k, r_k , and ϕ'_k have uniformly bounded second moments, it can be seen that for some constant $c_1 > 0, \forall k \geq 0$,

$$\mathbb{E}[\|M_{k+1}\|^2 | \mathcal{F}(k)] \leq c_1(1 + \|\omega_k\|^2 + \|\mathbf{u}_k\|^2 + \|\boldsymbol{\theta}_k\|^2).$$

Now Assumptions (A1) and (A2) of (Borkar and Meyn 2000) are verified. Furthermore, Assumptions (TS) of (Borkar and Meyn 2000) is satisfied by our conditions on the step-size sequences $\alpha_k, \zeta_k, \beta_k$. Thus, by Theorem 2.2 of (Borkar and Meyn 2000) we obtain that $\|\omega_k - \omega^*\| \rightarrow 0$ almost surely as $k \rightarrow \infty$.

Consider now the second time scale recursion (12). Based on the above analysis, (12) can be rewritten as

$$\dot{\boldsymbol{\theta}}(t) = 0, \tag{A-6}$$

$$\dot{\mathbf{u}}(t) = \mathbb{E}[(\delta_t - \mathbb{E}[\delta_t|\mathbf{u}(t), \boldsymbol{\theta}(t)])\boldsymbol{\phi}_t|\boldsymbol{\theta}(t)] - \mathbf{C}\mathbf{u}(t). \quad (\text{A-7})$$

The ODE (A-6) suggests that $\boldsymbol{\theta}(t) \equiv \boldsymbol{\theta}$ (i.e., a time invariant parameter) when viewed from the second fast timescale. By the Hirsch lemma (Hirsch 1989), it follows that $\|\boldsymbol{\theta}_k - \boldsymbol{\theta}\| \rightarrow 0$ a.s. as $k \rightarrow \infty$ for some $\boldsymbol{\theta}$ that depends on the initial condition $\boldsymbol{\theta}_0$ of recursion (11).

Consider now the recursion (12). Let $N_{k+1} = ((\delta_k - \mathbb{E}[\delta_k]) - \boldsymbol{\phi}_k \boldsymbol{\phi}_k^\top \mathbf{u}_k) - \mathbb{E}[(\delta_k - \mathbb{E}[\delta_k]) - \boldsymbol{\phi}_k \boldsymbol{\phi}_k^\top \mathbf{u}_k]|\mathcal{I}(k)]$, where $\mathcal{I}(k) = \sigma(\mathbf{u}_l, \boldsymbol{\theta}_l, l \leq k; \boldsymbol{\phi}_s, \boldsymbol{\phi}'_s, r_s, s < k)$, $k \geq 1$ are the sigma fields generated by $\mathbf{u}_0, \boldsymbol{\theta}_0, \mathbf{u}_{l+1}, \boldsymbol{\theta}_{l+1}, \boldsymbol{\phi}_l, \boldsymbol{\phi}'_l, 0 \leq l < k$. It is easy to verify that $N_{k+1}, k \geq 0$ are integrable random variables that satisfy $\mathbb{E}[N_{k+1}|\mathcal{I}(k)] = 0, \forall k \geq 0$. Because $\boldsymbol{\phi}_k, r_k$, and $\boldsymbol{\phi}'_k$ have uniformly bounded second moments, it can be seen that for some constant $c_2 > 0, \forall k \geq 0$,

$$\mathbb{E}[\|N_{k+1}\|^2|\mathcal{I}(k)] \leq c_2(1 + \|\mathbf{u}_k\|^2 + \|\boldsymbol{\theta}_k\|^2).$$

Because $\boldsymbol{\theta}(t) \equiv \boldsymbol{\theta}$ from (A-6), the ODE pair (A-6)-(A-7) can be written as

$$\dot{\mathbf{u}}(t) = \mathbb{E}[(\delta_t - \mathbb{E}[\delta_t|\boldsymbol{\theta}])\boldsymbol{\phi}_t|\boldsymbol{\theta}] - \mathbf{C}\mathbf{u}(t). \quad (\text{A-8})$$

Now consider the function $h(\mathbf{u}) = \mathbb{E}[\delta_t - \mathbb{E}[\delta_t|\boldsymbol{\theta}]]\boldsymbol{\phi}|\boldsymbol{\theta}] - \mathbf{C}\mathbf{u}$, i.e., the driving vector field of the ODE (A-8). For (A-8), $\mathbf{u}^* = \mathbf{C}^{-1}\mathbb{E}[(\delta - \mathbb{E}[\delta|\boldsymbol{\theta}])\boldsymbol{\phi}|\boldsymbol{\theta}]$ is the unique globally asymptotically stable equilibrium. Let $h_\infty(\mathbf{u}) = -\mathbf{C}\mathbf{u}$. For the ODE

$$\dot{\mathbf{u}}(t) = h_\infty(\mathbf{u}(t)) = -\mathbf{C}\mathbf{u}(t), \quad (\text{A-9})$$

the origin of (A-9) is a globally asymptotically stable equilibrium because \mathbf{C} is a positive definite matrix (because it is non-negative definite and nonsingular). Now Assumptions (A1) and (A2) of (Borkar and Meyn 2000) are verified. Furthermore, Assumptions (TS) of (Borkar and Meyn 2000) is satisfied by our conditions on the step-size sequences $\alpha_k, \zeta_k, \beta_k$. Thus, by Theorem 2.2 of (Borkar and Meyn 2000) we obtain that $\|\mathbf{u}_k - \mathbf{u}^*\| \rightarrow 0$ almost surely as $k \rightarrow \infty$.

Consider now the slower timescale recursion (11). In the light of the above, (11) can be rewritten as

$$\boldsymbol{\theta}_{k+1} \leftarrow \boldsymbol{\theta}_k + \alpha_k(\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}_k - \alpha_k\gamma\boldsymbol{\phi}'_k(\boldsymbol{\phi}_k^\top \mathbf{C}^{-1}\mathbb{E}[(\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}|\boldsymbol{\theta}_k]). \quad (\text{A-10})$$

Let $\mathcal{G}(k) = \sigma(\boldsymbol{\theta}_l, l \leq k; \boldsymbol{\phi}_s, \boldsymbol{\phi}'_s, r_s, s < k)$, $k \geq 1$ be the sigma fields generated by $\boldsymbol{\theta}_0, \boldsymbol{\theta}_{l+1}, \boldsymbol{\phi}_l, \boldsymbol{\phi}'_l, 0 \leq l < k$. Let

$$\begin{aligned} Z_{k+1} &= ((\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}_k - \gamma\boldsymbol{\phi}'_k\boldsymbol{\phi}_k^\top \mathbf{C}^{-1}\mathbb{E}[(\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}|\boldsymbol{\theta}_k]) \\ &\quad - \mathbb{E}[(\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}_k - \gamma\boldsymbol{\phi}'_k\boldsymbol{\phi}_k^\top \mathbf{C}^{-1}\mathbb{E}[(\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}|\boldsymbol{\theta}_k])|\mathcal{G}(k)] \\ &= ((\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}_k - \gamma\boldsymbol{\phi}'_k\boldsymbol{\phi}_k^\top \mathbf{C}^{-1}\mathbb{E}[(\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}|\boldsymbol{\theta}_k]) \\ &\quad - \mathbb{E}[(\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}_k|\boldsymbol{\theta}_k] - \gamma\mathbb{E}[\boldsymbol{\phi}'_k\boldsymbol{\phi}_k^\top]\mathbf{C}^{-1}\mathbb{E}[(\delta_k - \mathbb{E}[\delta_k|\boldsymbol{\theta}_k])\boldsymbol{\phi}_k|\boldsymbol{\theta}_k]). \end{aligned}$$

It is easy to see that $Z_k, k \geq 0$ are integrable random variables and $\mathbb{E}[Z_{k+1}|\mathcal{G}(k)] = 0, \forall k \geq 0$. Further,

$$\mathbb{E}[\|Z_{k+1}\|^2|\mathcal{G}(k)] \leq c_3(1 + \|\boldsymbol{\theta}_k\|^2), k \geq 0$$

for some constant $c_3 \geq 0$, again because $\boldsymbol{\phi}_k, r_k$, and $\boldsymbol{\phi}'_k$ have uniformly bounded second moments, it can be seen that for some constant.

Consider now the following ODE associated with (11):

$$\dot{\boldsymbol{\theta}}(t) = (\mathbf{I} - \mathbb{E}[\gamma\boldsymbol{\phi}'\boldsymbol{\phi}^\top]\mathbf{C}^{-1})\mathbb{E}[(\delta - \mathbb{E}[\delta|\boldsymbol{\theta}(t)])\boldsymbol{\phi}|\boldsymbol{\theta}(t)]. \quad (\text{A-11})$$

Let

$$\begin{aligned} \vec{h}(\boldsymbol{\theta}(t)) &= (\mathbf{I} - \mathbb{E}[\gamma\boldsymbol{\phi}'\boldsymbol{\phi}^\top]\mathbf{C}^{-1})\mathbb{E}[(\delta - \mathbb{E}[\delta|\boldsymbol{\theta}(t)])\boldsymbol{\phi}|\boldsymbol{\theta}(t)] \\ &= (\mathbf{C} - \mathbb{E}[\gamma\boldsymbol{\phi}'\boldsymbol{\phi}^\top])\mathbf{C}^{-1}\mathbb{E}[(\delta - \mathbb{E}[\delta|\boldsymbol{\theta}(t)])\boldsymbol{\phi}|\boldsymbol{\theta}(t)] \\ &= (\mathbb{E}[\boldsymbol{\phi}\boldsymbol{\phi}^\top] - \mathbb{E}[\gamma\boldsymbol{\phi}'\boldsymbol{\phi}^\top])\mathbf{C}^{-1}\mathbb{E}[(\delta - \mathbb{E}[\delta|\boldsymbol{\theta}(t)])\boldsymbol{\phi}|\boldsymbol{\theta}(t)] \\ &= \mathbf{A}^\top \mathbf{C}^{-1}(-\mathbf{A}\boldsymbol{\theta}(t) + \mathbf{b}), \end{aligned}$$

because $\mathbb{E}[(\delta - \mathbb{E}[\delta|\boldsymbol{\theta}(t)])\boldsymbol{\phi}|\boldsymbol{\theta}(t)] = -\mathbf{A}\boldsymbol{\theta}(t) + \mathbf{b}$, where $\mathbf{A} = \text{Cov}(\boldsymbol{\phi}, \boldsymbol{\phi} - \gamma\boldsymbol{\phi}')$, $\mathbf{b} = \text{Cov}(r, \boldsymbol{\phi})$, and $\mathbf{C} = \mathbb{E}[\boldsymbol{\phi}\boldsymbol{\phi}^\top]$

Therefore, $\boldsymbol{\theta}^* = \mathbf{A}^{-1}\mathbf{b}$ can be seen to be the unique globally asymptotically stable equilibrium for ODE (A-11). Let $\vec{h}_\infty(\boldsymbol{\theta}) = \lim_{r \rightarrow \infty} \frac{\vec{h}(r\boldsymbol{\theta})}{r}$. Then $\vec{h}_\infty(\boldsymbol{\theta}) = -\mathbf{A}^\top \mathbf{C}^{-1}\mathbf{A}\boldsymbol{\theta}$ is well-defined. Consider now the ODE

$$\dot{\boldsymbol{\theta}}(t) = -\mathbf{A}^\top \mathbf{C}^{-1}\mathbf{A}\boldsymbol{\theta}(t). \quad (\text{A-12})$$

Because \mathbf{C}^{-1} is positive definite and \mathbf{A} has full rank (as it is nonsingular by assumption), the matrix $\mathbf{A}^\top \mathbf{C}^{-1}\mathbf{A}$ is also positive definite. The ODE (A-12) has the origin as its unique globally asymptotically stable equilibrium. Thus, the assumption (A1) and (A2) are verified.

The proof is given above. In the fastest time scale, the parameter w converges to $\mathbb{E}[\delta|\mathbf{u}_k, \boldsymbol{\theta}_k]$. In the second fast time scale, the parameter u converges to $\mathbf{C}^{-1}\mathbb{E}[(\delta - \mathbb{E}[\delta|\boldsymbol{\theta}_k])\boldsymbol{\phi}|\boldsymbol{\theta}_k]$. In the slower time scale, the parameter $\boldsymbol{\theta}$ converges to $\mathbf{A}^{-1}\mathbf{b}$. \square

A.2 Proof of Theorem 3

Proof. The proof of VMETD's convergence is also based on Borkar's Theorem for general stochastic approximation recursions with two time scales (Borkar 1997).

The VMTD's solution is $\theta_{\text{VMETD}} = \mathbf{A}_{\text{VMETD}}^{-1} \mathbf{b}_{\text{VMETD}}$.

First, note that recursion (19) can be rewritten as

$$\theta_{k+1} \leftarrow \theta_k + \beta_k \xi(k),$$

where

$$\xi(k) = \frac{\alpha_k}{\beta_k} (F_k \rho_k \delta_k - \omega_{k+1}) \phi_k$$

Due to the settings of step-size schedule $\alpha_k = o(\beta_k)$, $\xi(k) \rightarrow 0$ almost surely as $k \rightarrow \infty$. That is the increments in iteration (20) are uniformly larger than those in (19), thus (20) is the faster recursion. Along the faster time scale, iterations of (20) and (19) are associated to ODEs system as follows:

$$\dot{\theta}(t) = 0, \quad (\text{A-13})$$

$$\dot{\omega}(t) = \mathbb{E}_\mu[F_t \rho_t \delta_t | \theta(t)] - \omega(t). \quad (\text{A-14})$$

Based on the ODE (A-13), $\theta(t) \equiv \theta$ when viewed from the faster timescale. By the Hirsch lemma (Hirsch 1989), it follows that $\|\theta_k - \theta\| \rightarrow 0$ a.s. as $k \rightarrow \infty$ for some θ that depends on the initial condition θ_0 of recursion (19). Thus, the ODE pair (A-13)-(A-14) can be written as

$$\dot{\omega}(t) = \mathbb{E}_\mu[F_t \rho_t \delta_t | \theta] - \omega(t). \quad (\text{A-15})$$

Consider the function $h(\omega) = \mathbb{E}_\mu[F \rho \delta | \theta] - \omega$, i.e., the driving vector field of the ODE (A-15). It is easy to find that the function h is Lipschitz with coefficient -1 . Let $h_\infty(\cdot)$ be the function defined by $h_\infty(\omega) = \lim_{x \rightarrow \infty} \frac{h(x\omega)}{x}$. Then $h_\infty(\omega) = -\omega$, is well-defined. For (A-15), $\omega^* = \mathbb{E}_\mu[F \rho \delta | \theta]$ is the unique globally asymptotically stable equilibrium. For the ODE

$$\dot{\omega}(t) = h_\infty(\omega(t)) = -\omega(t), \quad (\text{A-16})$$

apply $\bar{V}(\omega) = (-\omega)^\top (-\omega)/2$ as its associated strict Liapunov function. Then, the origin of (A-16) is a globally asymptotically stable equilibrium.

Consider now the recursion (20). Let $M_{k+1} = (F_k \rho_k \delta_k - \omega_k) - \mathbb{E}_\mu[(F_k \rho_k \delta_k - \omega_k) | \mathcal{F}(k)]$, where $\mathcal{F}(k) = \sigma(\omega_l, \theta_l, l \leq k; \phi_s, \phi'_s, r_s, s < k)$, $k \geq 1$ are the sigma fields generated by $\omega_0, \theta_0, \omega_{l+1}, \theta_{l+1}, \phi_l, \phi'_l$, $0 \leq l < k$. It is easy to verify that $M_{k+1}, k \geq 0$ are integrable random variables that satisfy $\mathbb{E}[M_{k+1} | \mathcal{F}(k)] = 0$, $\forall k \geq 0$. Because ϕ_k, r_k , and ϕ'_k have uniformly bounded second moments, it can be seen that for some constant $c_1 > 0$, $\forall k \geq 0$,

$$\mathbb{E}[\|M_{k+1}\|^2 | \mathcal{F}(k)] \leq c_1(1 + \|\omega_k\|^2 + \|\theta_k\|^2).$$

Now Assumptions (A1) and (A2) of (Borkar and Meyn 2000) are verified. Furthermore, Assumptions (TS) of (Borkar and Meyn 2000) is satisfied by our conditions on the step-size sequences α_k, β_k . Thus, by Theorem 2.2 of (Borkar and Meyn 2000) we obtain that $\|\omega_k - \omega^*\| \rightarrow 0$ almost surely as $k \rightarrow \infty$.

Consider now the slower time scale recursion (19). Based on the above analysis, (19) can be rewritten as

$$\begin{aligned} \theta_{k+1} &\leftarrow \theta_k + \alpha_k (F_k \rho_k \delta_k - \omega_k) \phi_k - \alpha_k \omega_{k+1} \phi_k \\ &= \theta_k + \alpha_k (F_k \rho_k \delta_k - \mathbb{E}_\mu[F_k \rho_k \delta_k | \theta_k]) \phi_k \\ &= \theta_k + \alpha_k F_k \rho_k (R_{k+1} + \gamma \theta_k^\top \phi_{k+1} - \theta_k^\top \phi_k) \phi_k - \alpha_k \mathbb{E}_\mu[F_k \rho_k \delta_k] \phi_k \\ &= \theta_k + \alpha_k \underbrace{\{(F_k \rho_k R_{k+1} - \mathbb{E}_\mu[F_k \rho_k R_{k+1}]) \phi_k\}}_{\mathbf{b}_{\text{VMETD},k}} - \underbrace{\{(F_k \rho_k \phi_k (\phi_k - \gamma \phi_{k+1})^\top - \phi_k \mathbb{E}_\mu[F_k \rho_k (\phi_k - \gamma \phi_{k+1})]^\top) \theta_k\}}_{\mathbf{A}_{\text{VMETD},k}} \end{aligned}$$

Let $\mathcal{G}(k) = \sigma(\theta_l, l \leq k; \phi_s, \phi'_s, r_s, s < k)$, $k \geq 1$ be the sigma fields generated by $\theta_0, \theta_{l+1}, \phi_l, \phi'_l$, $0 \leq l < k$. Let $Z_{k+1} = Y_k - \mathbb{E}[Y_k | \mathcal{G}(k)]$, where

$$Y_k = (F_k \rho_k \delta_k - \mathbb{E}_\mu[F_k \rho_k \delta_k | \theta_k]) \phi_k.$$

Consequently,

$$\begin{aligned} \mathbb{E}_\mu[Y_k | \mathcal{G}(k)] &= \mathbb{E}_\mu[(F_k \rho_k \delta_k - \mathbb{E}_\mu[F_k \rho_k \delta_k | \theta_k]) \phi_k | \mathcal{G}(k)] \\ &= \mathbb{E}_\mu[F_k \rho_k \delta_k \phi_k | \theta_k] - \mathbb{E}_\mu[\mathbb{E}_\mu[F_k \rho_k \delta_k | \theta_k] \phi_k] \\ &= \mathbb{E}_\mu[F_k \rho_k \delta_k \phi_k | \theta_k] - \mathbb{E}_\mu[F_k \rho_k \delta_k | \theta_k] \mathbb{E}_\mu[\phi_k] \\ &= \text{Cov}(F_k \rho_k \delta_k | \theta_k, \phi_k), \end{aligned}$$

where $\text{Cov}(\cdot, \cdot)$ is a covariance operator.

Thus,

$$Z_{k+1} = (F_k \rho_k \delta_k - \mathbb{E}[\delta_k | \theta_k]) \phi_k - \text{Cov}(F_k \rho_k \delta_k | \theta_k, \phi_k).$$

It is easy to verify that $Z_{k+1}, k \geq 0$ are integrable random variables that satisfy $\mathbb{E}[Z_{k+1}|\mathcal{G}(k)] = 0, \forall k \geq 0$. Also, because ϕ_k, r_k , and ϕ'_k have uniformly bounded second moments, it can be seen that for some constant $c_2 > 0, \forall k \geq 0$,

$$\mathbb{E}[||Z_{k+1}||^2|\mathcal{G}(k)] \leq c_2(1 + ||\theta_k||^2).$$

Consider now the following ODE associated with (19):

$$\dot{\theta}(t) = -\mathbf{A}_{\text{VMETD}}\theta(t) + \mathbf{b}_{\text{VMETD}}. \quad (\text{A-17})$$

$$\begin{aligned} \mathbf{A}_{\text{VMETD}} &= \lim_{k \rightarrow \infty} \mathbb{E}[\mathbf{A}_{\text{VMETD},k}] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}_\mu[F_k \rho_k \phi_k (\phi_k - \gamma \phi_{k+1})^\top] - \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k] \mathbb{E}_\mu[F_k \rho_k (\phi_k - \gamma \phi_{k+1})]^\top \\ &= \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k F_k \rho_k (\phi_k - \gamma \phi_{k+1})^\top] - \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k] \mathbb{E}_\mu[F_k \rho_k (\phi_k - \gamma \phi_{k+1})]^\top \\ &= \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k F_k \rho_k (\phi_k - \gamma \phi_{k+1})^\top] - \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k] \lim_{k \rightarrow \infty} \mathbb{E}_\mu[F_k \rho_k (\phi_k - \gamma \phi_{k+1})]^\top \\ &= \sum_s f(s) \phi(s) (\phi(s) - \gamma \sum_{s'} [\mathbf{P}_\pi]_{ss'} \phi(s'))^\top - \sum_s d_\mu(s) \phi(s) * \sum_s f(s) (\phi(s) - \gamma \sum_{s'} [\mathbf{P}_\pi]_{ss'} \phi(s'))^\top \quad (\text{A-18}) \\ &= \Phi^\top \mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) \Phi - \Phi^\top \mathbf{d}_\mu \mathbf{f}^\top (\mathbf{I} - \gamma \mathbf{P}_\pi) \Phi \\ &= \Phi^\top (\mathbf{F} - \mathbf{d}_\mu \mathbf{f}^\top) (\mathbf{I} - \gamma \mathbf{P}_\pi) \Phi \\ &= \Phi^\top (\mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) - \mathbf{d}_\mu \mathbf{f}^\top (\mathbf{I} - \gamma \mathbf{P}_\pi)) \Phi \\ &= \Phi^\top (\mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) - \mathbf{d}_\mu \mathbf{d}_\mu^\top) \Phi \end{aligned}$$

$$\begin{aligned} \mathbf{b}_{\text{VMETD}} &= \lim_{k \rightarrow \infty} \mathbb{E}[\mathbf{b}_{\text{VMETD},k}] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}_\mu[F_k \rho_k R_{k+1} \phi_k] - \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k] \mathbb{E}_\mu[F_k \rho_k R_{k+1}] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k F_k \rho_k R_{k+1}] - \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k] \mathbb{E}_\mu[F_k \rho_k R_{k+1}] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k F_k \rho_k R_{k+1}] - \lim_{k \rightarrow \infty} \mathbb{E}_\mu[\phi_k] \lim_{k \rightarrow \infty} \mathbb{E}_\mu[F_k \rho_k R_{k+1}] \quad (\text{A-19}) \\ &= \sum_s f(s) \phi(s) r_\pi - \sum_s d_\mu(s) \phi(s) * \sum_s f(s) r_\pi \\ &= \Phi^\top (\mathbf{F} - \mathbf{d}_\mu \mathbf{d}_\mu^\top) \mathbf{r}_\pi \end{aligned}$$

Let $\vec{h}(\theta(t))$ be the driving vector field of the ODE (A-17).

$$\vec{h}(\theta(t)) = -\mathbf{A}_{\text{VMETD}}\theta(t) + \mathbf{b}_{\text{VMETD}}.$$

An $\Phi^\top \mathbf{X} \Phi$ matrix of this form will be positive definite whenever the matrix \mathbf{X} is positive definite. Any matrix \mathbf{X} is positive definite if and only if the symmetric matrix $\mathbf{S} = \mathbf{X} + \mathbf{X}^\top$ is positive definite. Any symmetric real matrix \mathbf{S} is positive definite if the absolute values of its diagonal entries are greater than the sum of the absolute values of the corresponding off-diagonal entries (Sutton, Mahmood, and White 2016).

$$\begin{aligned} (\mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) - \mathbf{d}_\mu \mathbf{d}_\mu^\top) \mathbf{1} &= \mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) \mathbf{1} - \mathbf{d}_\mu \mathbf{d}_\mu^\top \mathbf{1} \\ &= \mathbf{F}(\mathbf{1} - \gamma \mathbf{P}_\pi \mathbf{1}) - \mathbf{d}_\mu \mathbf{d}_\mu^\top \mathbf{1} \\ &= (1 - \gamma) \mathbf{F} \mathbf{1} - \mathbf{d}_\mu \mathbf{d}_\mu^\top \mathbf{1} \\ &= (1 - \gamma) \mathbf{f} - \mathbf{d}_\mu \mathbf{d}_\mu^\top \mathbf{1} \\ &= (1 - \gamma) \mathbf{f} - \mathbf{d}_\mu \\ &= (1 - \gamma) (\mathbf{I} - \gamma \mathbf{P}_\pi^\top)^{-1} \mathbf{d}_\mu - \mathbf{d}_\mu \quad (\text{A-20}) \\ &= (1 - \gamma) [(\mathbf{I} - \gamma \mathbf{P}_\pi^\top)^{-1} - \mathbf{I}] \mathbf{d}_\mu \\ &= (1 - \gamma) \left[\sum_{t=0}^{\infty} (\gamma \mathbf{P}_\pi^\top)^t - \mathbf{I} \right] \mathbf{d}_\mu \\ &= (1 - \gamma) \left[\sum_{t=1}^{\infty} (\gamma \mathbf{P}_\pi^\top)^t \right] \mathbf{d}_\mu > 0 \end{aligned}$$

$$\begin{aligned}
\mathbf{1}^\top (\mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) - \mathbf{d}_\mu \mathbf{d}_\mu^\top) &= \mathbf{1}^\top \mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) - \mathbf{1}^\top \mathbf{d}_\mu \mathbf{d}_\mu^\top \\
&= \mathbf{d}_\mu^\top - \mathbf{1}^\top \mathbf{d}_\mu \mathbf{d}_\mu^\top \\
&= \mathbf{d}_\mu^\top - \mathbf{d}_\mu^\top \\
&= 0
\end{aligned} \tag{A-21}$$

(A-20) and (A-21) show that the matrix $\mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) - \mathbf{d}_\mu \mathbf{d}_\mu^\top$ of diagonal entries are positive and its off-diagonal entries are negative. So its each row sum plus the corresponding column sum is positive. So $\mathbf{A}_{\text{VMETD}}$ is positive definite.

Therefore, $\theta^* = \mathbf{A}_{\text{VMETD}}^{-1} \mathbf{b}_{\text{VMETD}}$ can be seen to be the unique globally asymptotically stable equilibrium for ODE (A-17). Let $\vec{h}_\infty(\theta) = \lim_{r \rightarrow \infty} \frac{h(r\theta)}{r}$. Then $\vec{h}_\infty(\theta) = -\mathbf{A}_{\text{VMETD}}\theta$ is well-defined. Consider now the ODE

$$\dot{\theta}(t) = -\mathbf{A}_{\text{VMETD}}\theta(t). \tag{A-22}$$

The ODE (A-22) has the origin as its unique globally asymptotically stable equilibrium. Thus, the assumption (A1) and (A2) are verified. \square

B Mathematical Analysis

Table 1 shows the key matrices of various algorithms. Table 2 shows minimum eigenvalues $\frac{1}{2}\lambda_{\min}(\mathbf{A} + \mathbf{A}^\top)$ of various algorithms on several examples.

Table 1: Key matrices of various algorithms.

Algorithm	Key matrix \mathbf{A}	Positive definite	\mathbf{b}
On-policy TD	$\Phi^\top \mathbf{D}_\pi (\mathbf{I} - \gamma \mathbf{P}_\pi) \Phi$	✓	$\mathbf{b}_{\text{on}} = \Phi^\top \mathbf{D}_\pi \mathbf{r}_\pi$
On-policy VMTD	$\Phi^\top (\mathbf{D}_\pi - \mathbf{d}_\pi \mathbf{d}_\pi^\top) (\mathbf{I} - \gamma \mathbf{P}_\pi) \Phi$	✓	$\Phi^\top (\mathbf{D}_\pi - \mathbf{d}_\pi \mathbf{d}_\pi^\top) \mathbf{r}_\pi$
Off-policy TD	$\mathbf{A}_{\text{off}} = \Phi^\top \mathbf{D}_\mu (\mathbf{I} - \gamma \mathbf{P}_\pi) \Phi$	×	$\mathbf{b}_{\text{off}} = \Phi^\top \mathbf{D}_\mu \mathbf{r}_\pi$
TDC	$\mathbf{A}_{\text{off}}^\top \mathbf{C}^{-1} \mathbf{A}_{\text{off}}$	✓	$\mathbf{A}_{\text{off}}^\top \mathbf{C}^{-1} \mathbf{b}_{\text{off}}$
ETD	$\Phi^\top \mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) \Phi$	✓	$\Phi^\top \mathbf{F} \mathbf{r}_\pi$
Off-policy VMTD	$\mathbf{A}_{\text{VM}} = \Phi^\top (\mathbf{D}_\mu - \mathbf{d}_\mu \mathbf{d}_\mu^\top) (\mathbf{I} - \gamma \mathbf{P}_\pi) \Phi$	×	$\mathbf{b}_{\text{VM}} = \Phi^\top (\mathbf{D}_\mu - \mathbf{d}_\mu \mathbf{d}_\mu^\top) \mathbf{r}_\pi$
VMTDC	$\mathbf{A}_{\text{VM}}^\top \mathbf{C}^{-1} \mathbf{A}_{\text{VM}}$	✓	$\mathbf{A}_{\text{VM}}^\top \mathbf{C}^{-1} \mathbf{b}_{\text{VM}}$
VMETD	$\Phi^\top (\mathbf{F}(\mathbf{I} - \gamma \mathbf{P}_\pi) - \mathbf{d}_\mu \mathbf{d}_\mu^\top) \Phi$	✓	$\Phi^\top (\mathbf{F} - \mathbf{d}_\mu \mathbf{d}_\mu^\top) \mathbf{r}_\pi$

Table 2: Minimum eigenvalues $\frac{1}{2}\lambda_{\min}(\mathbf{A} + \mathbf{A}^\top)$ of various algorithms on several examples.

Algorithm	2-state	Baird's	Random Walk			Boyan Chain
			Tabular	Inverted	Dependent	
TD	−0.2	−0.791	0.018	0.017	0.07	0.024
VMTD						
TDC	0.016	−0.002	0.002	0.007	0.011	0.002
VMTDC						
ETD	3.4	$-2.82e - 16$	0.195	0.165	0.76	0.245
VMETD						

C Experimental details

The feature matrices corresponding to three random walks are shown below respectively:

$$\Phi_{\text{tabular}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Table 3: Baird’s Counterexample.

Algorithm	Baird’s Counterexample	
	Two State	Seven State
TD	[-0.2]	[-7.91e-01, 1.07e-16, 5.71e-01, 5.71e-01, 5.71e-01, 5.71e-01, 5.71e-01, 7.48e-01]
VMTD	[0.25]	[-2.25e-01, -3.45e-17, 5.71e-01, 5.71e-01, 5.71e-01, 5.71e-01, 5.71e-01, 2.87e+00]
TDC	[0.016]	[-0.0024, 0.0078, 0.5714, 0.5714, 0.5714, 0.5714, 0.5729, 1.7682]
VMTDC	[0.025]	[-1.55e-03, -3.15e-04, 3.56e-01, 5.47e-01, 5.70e-01, 5.72e-01, 5.83e-01, 5.98e-01]
ETD	[3.4]	[-2.82e-16, 5.71e-01, 5.71e-01, 5.71e-01, 5.71e-01, 5.71e-01, 7.40e-01, 3.72e+00]
VMETD	[1.15]	[-2.25e-01, -3.45e-17, 5.71e-01, 5.71e-01, 5.71e-01, 5.71e-01, 5.71e-01, 2.86e+00]

Table 4: Random Walk.

Algorithm	Random Walk		
	Tabular	Inverted	Dependent
TD	[0.018,0.074,0.183,0.260,0.464]	[0.017,0.044,0.065,0.083,0.116]	[0.070,0.113,0.138]
VMTD	[1.62e-17,0.073,0.180,0.260,0.463]	[-2.07e-17,0.018,0.045,0.065,0.115]	[0.022,0.115,0.116]
TDC	[0.002,0.046,0.240,0.364,0.769]	[0.007,0.012,0.060,0.091,0.192]	[0.011,0.065,0.182]
VMTDC	[6.74e-17,0.044,0.240,0.364,0.769]	[-7.248e-18,0.011,0.060,0.091,0.192]	[0.0008,0.062,0.182]
ETD	[0.195,0.669,1.712,2.765,4.660]	[0.165,0.420,0.678,0.820,1.168]	[0.760,1.084,1.394]
VMETD	[-3.40e-04,0.664,1.69,2.76,4.65]	[-0.001,0.167,0.423,0.689,1.163]	[0.221,1.043,1.293]

$$\Phi_{inverted} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\Phi_{dependent} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix}$$

Three random walk experiments: the α values for all algorithms are in the range of $\{0.008, 0.015, 0.03, 0.06, 0.12, 0.25, 0.5\}$. For the TDC algorithm, the range of the ratio $\frac{\zeta}{\alpha}$ is $\{\frac{1}{512}, \frac{1}{256}, \frac{1}{128}, \frac{1}{64}, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2\}$. For the VMTD algorithm, the range of the ratio $\frac{\beta}{\alpha}$ is $\{\frac{1}{512}, \frac{1}{256}, \frac{1}{128}, \frac{1}{64}, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2\}$. It can be observed from the update formula of VMTDC that when ζ takes a very small value, the VMTDC update tends to be similar to VMTD update. Similarly, when β takes a very small value, the VMTDC update tends to be similar to TDC update. Through experiments, it was found that setting ζ to a small value makes VMTDC updates approach VMTD updates, resulting in better performance. Therefore, for the VMTDC algorithm, the range

Table 5: Boyan Chain.

Algorithm	eigenvalues
TD	[0.024495,0.054,0.065,0.135]
VMTD	[2.70e-18,0.053,0.065,0.135]
TDC	[0.002,0.058,0.067,0.153]
VMTDC	[-1.40e-18,0.057,0.067,0.153]
ETD	[0.245,0.540,0.647,1.352]
VMETD	[1.57e-17,0.529,0.647,1.352]

of $\frac{\beta}{\alpha}$ ratio is $\{\frac{1}{512}, \frac{1}{256}, \frac{1}{128}, \frac{1}{64}, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2\}$, and the range of ζ is $\{0.1, 0.01, 0.001, 0.0001, 0.00001\}$. The learning curves in Figure ?? correspond to the optimal parameters.

The feature matrix of 7-state version of Baird's off-policy counterexample is defined as follow:

$$\Phi_{Counter} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

7-state version of Baird's off-policy counterexample: for TD algorithm, α is set to 0.1. For the TDC algorithm, the range of α is $\{0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$, and the range of ζ is $\{0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5\}$. For the VMTD algorithm, the range of α is $\{0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$, and the range of β is $\{0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5\}$. Through experiments, it was found that setting ζ to a small value makes VMTDC updates approach VMTD updates, resulting in better performance. Therefore, for the VMTDC algorithm, The range of values for α and β is the same as that of VMTD and the range of ζ is $\{0.1, 0.01, 0.001, 0.0001, 0.00001\}$. The learning curves in Figure ?? correspond to the optimal parameters. For all policy evaluation experiments, each experiment is independently run 100 times.

For the four control experiments: The learning rates for each algorithm in all experiments are shown in Table 6. For all control experiments, each experiment is independently run 50 times.

Table 6: Learning rates (lr) of four control experiments.

algorithms(lr) \ envs	Maze	Cliff walking	Mountain Car	Acrobot
Sarsa(α)	0.1	0.1	0.1	0.1
GQ(0)(α, ζ)	0.1, 0.003	0.1, 0.004	0.1, 0.01	0.1, 0.01
VMSarsa(α, β)	0.1, 0.001	0.1, 1e-4	0.1, 1e-4	0.1, 1e-4
VMGQ(0)(α, ζ, β)	0.1, 0.001, 0.001	0.1, 0.005, 1e-4	0.1, 5e-4, 1e-4	0.1, 5e-4, 1e-4
AC(lr_{actor}, lr_{critic})	0.01, 0.1	0.01, 0.01	0.01, 0.05	0.01, 0.05
Q-learning(α)	0.1	0.1	0.1	0.1
VMQ(α, β)	0.1, 0.001	0.1, 1e-4	0.1, 1e-4	0.1, 1e-4

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